Summation of Series

Arithmetic Series

$$S_n = \frac{\mathrm{n}}{2}(2\mathrm{a} + (\mathrm{n} - 1)\mathrm{d})$$

This equation gives the value of the sum of a certain number of terms (n) in an arithmetic sequence. a is the first term of the sequence and d is the difference between terms.
An arithmetic sequence is a sequence where there is the same difference between each term. For example, 1 2 3 4… ($a = 1$ and $d = 1$) or 19 14 9 4…($a = 19, d = -5$)
The equation for any term in an arithmetic sequence is $U_n = a + (n-1)d$, so if $U_1 = a = 3$ and $d = 12$, then $U_7 = 3 + (7-1)12 = 75$

Proof

If $S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-3)d) + (a+(n-2)d) + (a+(n-1)d)$ (A)
Then $S_n = (a+(n-1)d) + (a+(n-2)d) + (a+(n-3)d) + \cdots + (a+2d) + (a+d) + a$ (B)
Comparing similarly highlighted terms, they add to $2a + (n-1)d$ and so
(A) + (B) = $2S_n = n(2a + (n-1)d)$
So $S_n = \frac{n}{2}(2a + (n-1)d)$

Geometric Series

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

This equation gives the value of the sum of a certain number of terms (n) in a geometric sequence. As before, a is the first term of the sequence, but r is the ratio between terms.
A geometric sequence is a sequence where there is the same ratio between terms. For example, 2 4 8 16 ($a = 2, r = 2$ – meaning you must multiply by 2 to get the next term) or 160 16 1.6 0.16 ($a = 160, r = \frac{1}{10}$)
The equation for any term in a geometric sequence is $U_n = ar^{(n-1)}$, so if $U_1 = a = 4$ and $r = 3$, then $U_8 = 4 \times 3^{(8-1)}$, $U_8 = 8748$

Proof

If $S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}$ (A)
then $rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n$ (B)
(A) – (B) = $S_n - rS_n = a - ar^n$ [as all other terms are cancelled out]
So $S_n(1-r) = a(1-r^n)$
Re-arranging gives $S_n = \frac{a(1-r^n)}{(1-r)}$

Sum to infinity

$$S_\infty = \frac{a}{1-r}$$

This expression can only be used for a geometric series that is convergent, meaning -1<r<1 and so adding up every term in the series to infinity will give a certain value.

For example, if $a = 4$ and $r = \frac{1}{2}$ we have the terms 4 2 1 ½ ¼ and so on.

The sum to infinity is given by $S_\infty = \frac{4}{1-\frac{1}{2}} = 8$ we can see that this could be true merely by considering the first five terms. $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} = 7\frac{3}{4}$ so we can see how this summation is approached 8.

Proof

Consider the equation

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

as $n \to \infty$[1], $r^n \to 0$ so effectively

$$S_\infty = \frac{a(1-0)}{(1-r)}$$

so

$$S_\infty = \frac{a}{1-r}$$

The reason for $r^n \to 0$, is if you consider $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \dots$ the result will get smaller and smaller and so closer and closer to 0.

References

Attwood, G. et al. (2017). *Edexcel A level Mathematics - Pure - Year 2*. London: Pearson Education. pp.63-73.

[1] This means "as n tends to infinity" or "as n approaches infinity".